E-day

-Open Problem

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Open Problem

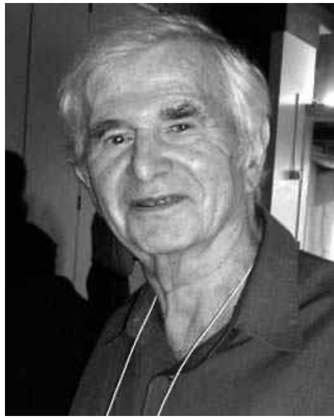
- Consider the Anytime-Valid T-Test likelihood ratio:

$$S^{(n)} = \frac{\int f_{\delta_1 \sigma, \sigma}(X^n) \left(\frac{1}{\sigma}\right) d\sigma}{\int f_{0,\sigma}(X^n) \left(\frac{1}{\sigma}\right) d\sigma} = \frac{g_{\delta_1}(V^n)}{g_0(V^n)}$$

- This is a test martingale, hence an e-process under $H_0 = \{F_{0,\sigma} : \sigma > 0\}$
- [Perez-Ortiz et al. *Annals*, 2024] For fixed n, it is also an e-variable under larger null

  $H_0' = \{F_{\mu,\sigma} : \sigma > 0, \mu \leq 0\}$ (relevant in 1-sided testing)
- Aaditya and Hongian asked: is $S^{(n)}$ also an e-process under H_0' ?
- **…embarrasingly, we don't know!!!**

A Paper You Must Read . (Period.)



Samuel Karlin

- Variation Diminishing Transformations: A Direct Approach to Total Positivity and Its Statistical Applications.
  L. Brown, I. Johnstone and K. B. MacGibbon, *JASA* 1981

- Don't get scared by the title, which with hindsight is most ill-chosen!

- Paper makes **Karlin**'s theory of the 1950/1960s about monotone likelihood ratios, stochastic dominance etc. much more accessible by **completely avoiding** the highly involved concept of 'total positivity', which is central in Karlin's treatment

Sign Changes

- Let $X \subset \mathbb{R}$ be an interval and $f: X \to \mathbb{R}$ be a continuous function.
- $S^{-}(f)$ stands for the number of sign changes of f

- Generalization to f defined on finite $X \subset \mathbb{R}$: let $x_1' \leq \cdots \leq x_n'$ be the ordered sequence of elements of X. Then $S^{-}(f)$ is the number of sign changes of $(f(x_1'), \ldots, f(x_n'))$, not counting 0s.

VR: Variation Reducing

Let $X \subset \mathbb{R}$, $\Theta \subset \mathbb{R}$ and $f\colon \Theta \times X \to \mathbb{R}_0^+$
(write $f_\theta(x)$ and think of it as density of X under some measure F_θ)
We say "f is VR_{n+1} on X with parameter Θ" if for all nonnegative measures ν on X and functions $g\colon X \to \mathbb{R}$ with $\int |g| d\,\nu > 0$, we have:

$$S^-(g) \leq n \Rightarrow S^-(\gamma) \leq S^-(g)$$

where $\gamma(\theta) \coloneqq \int f_\theta(x)\, g(x)\nu(dx)$.

If f_θ is indeed a probability measure relative to ν this means:
if the function g changes sign $m \leq n$ times when we vary x, then its expectation changes sign **at most** m times when we vary θ

Example:,
Exponential Families are $\boldsymbol{VR_2}$

- $f_\theta(x) = \frac{\exp(\theta \cdot x)}{\int \exp(\theta \cdot x) d\rho(x)}$ with Θ the natural parameter space defines an (arbitrary) 1-dim exp family; P_θ has density f_θ relative to ρ

- It can be shown that any such f is VR_2. This immediately implies that for any monotone increasing function g, we have that $E_{P_\theta}[g(X)]$ is an monotone increasing function in θ

- …but this property is also known as stochastic dominance!
- …and it is well-known that for 1-dim exp families, P_θ stochastically dominates $P_{\theta'}$ whenever $\theta > \theta'$

$\boldsymbol{VR_2}$ is stochastic dominance!

- For many other families besides exponential families, it can also be shown that they are VR_2. For example, the noncentral t- and $\boldsymbol{\chi^2}$-families are $\boldsymbol{VR_2}$ as well.

- In fact stochastic dominance is equivalent to VR_2

$\boldsymbol{VR_\infty}$

- 1-dim exponential families, noncentral χ^2 and noncentral F families are even VR_∞ , which abbreviates "VR_{n+1} for all n"

- Taking expectation under θ of any function which changes sign n times and varying θ gives you a function that changes sign at most n times

- There are many more families with finite or infinite VR properties.

- **But how to prove this for any given family?**

Central Theorem (Deep)

- Theorem 3.1 (going back to Karlin's works of the 1950s and 1960s):

f is VR_{n+1} on X with parameter Θ if and only if

for every pair of finite subsets $X' \subset X$, $\Theta' \subset \Theta$, both with $n+1$ elements, f is VR_{n+1} on X' with parameter Θ'

The second, "finite" form is often quite easy to check: involves only summation, no need to perform integrals relative to all measures on interval X !

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- [Perez-Ortiz et al. *Annals*, 2024] For fixed n, it is an e-variable under larger null

$$H_0' = \{F_{\delta\sigma,\sigma} : \sigma > 0, \delta \ \leq 0\}$$

This null sometimes more relevant (one-sided testing)

- Aaditya and Hongian asked: is $S^{(n)}$ also an e-process under H_0' ?
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Proposition: $S^{(n)}$ is e-variable, i.e. $\forall\delta \leq 0: u(\delta) \leq 1$ with $u(\delta) := \mathbf{E}_{H_\delta}[S^{(n)}]$

- VR-Reformulation of proof: $u(0) = 1$ (trivially by cancellation)
- LR is monotone increasing in T_n. Since 1-parameter family $h_\delta(T_n)$ is VR_2, so is $u(\delta)$. The result follows!

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To prove $\left(S^{(n)}\right)_n$ is e-process, sufficient to prove:

$\forall n, v^n, \delta \leq 0$: $u(\delta)|v^n \leq 1$ with $u(\delta|v^n) \coloneqq \mathbf{E}_{H_\delta}\left[\frac{h_{\delta_1}(T_n|v^{n-1})}{h_0(T_n|v^{n-1})} |V^{n-1} = v^{n-1}\right]$

Conjecture: for any constant C, (a) conditional LR inside expectation has at most 1 extremum, (b) conditional densities are VR_3. If (a)+(b) true, then $u(\delta|v^n)$ has at most 1 extremum, and the result seems provable